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Fourier Asymptotics of Fractal Measures

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IN MEMORY OF A. S. BESICOVITCH

A measure μ on $\mathbb{R}^n$ will be called *locally uniformly α-dimensional* if $\mu(B_r(x)) \leqslant cr^\alpha$ for all $r \leqslant 1$ and all x, where $B_r(x)$ denotes the ball of radius r about x. For $f \in L^2(d\mu)$, the measure $f\,d\mu$ is in $\mathscr{S}'$ so $(f\,d\mu)^\wedge$ is well-defined. We show it is locally in L^2 and

$$\sup_{r \geqslant 1} r^{\alpha - n} \int_{B_r(y)} |(f\,d\mu)^\wedge(\xi)|^2\,d\xi \leqslant c\,\|f\|_2.$$

Under additional hypotheses we show that

$$\lim_{r \to \infty} r^{\alpha - n} \int_{B_r(y)} |(f\,d\mu)^\wedge(\xi)|^2\,d\xi$$

is comparable in size to $\|f\|_2^2$. A number of other related results are established. The special case when α is an integer and μ is the surface measure on a C^1 manifold was treated by S. Agmon and L. Hörmander (*J. Analyse Math.* **30**, 1976, 1–38).

1. INTRODUCTION

An almost periodic function $F(\xi) = \sum c_j e^{i\xi, a_j}$ can be thought of as the Fourier transform $(f\,d\mu)^\wedge(\xi)$ where μ is the discrete measure $\mu = \sum \delta(x - a_j)$ and $f(a_j) = c_j$ is in $L^2(d\mu)$. The Parseval theorem for F says

$$\lim_{r \to \infty} r^{-n} \int_{B_r(y)} |(f\,d\mu)^\wedge(\xi)|^2\,d\xi = c \int |f|^2\,d\mu$$

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for any fixed y, the constant c being the measure of B_1. This can be thought of as the zero-dimensional case of our results. More generally, a well-known theorem of Wiener [W, Theorem 24, p. 146] states that if μ is any finite measure and $\mu = \sum c_j \delta(x - a_j) + \nu$, where ν is a continuous measure, then $\lim_{r \to \infty} r^{-n} \int_{B_r(y)} |(d\mu)^\wedge(\xi)|^2 \, d\xi = c \sum |c_j|^2$. From our point of view, this indicates that while all finite measures may be regarded as zero-dimensional, only the discrete part, which has a Radon–Nikodym derivative with respect to zero-dimensional Hausdorff measure, contributes to the formula.

At the other extreme, the Plancherel formula can be written

$$\lim_{r \to \infty} \int_{B_r(y)} |(f \, dx)^\wedge(\xi)|^2 \, d\xi = (2\pi)^n \int |f(x)|^2 \, dx.$$

This is the n-dimensional case of our results. A naive conjecture, "interpolating" between these results, would be that

$$\lim_{r \to \infty} r^{\alpha - n} \int_{B_r(y)} |(f \, d\mu)^\wedge(\xi)|^2 \, d\xi = c \int |f|^2 \, d\mu$$

for suitable α-dimensional measures μ. As it turns out, this conjecture is too strong, and the limit fails to exist in most cases. Nevertheless, the spirit of the conjecture is correct, and if we replace the limit by limsup then we do get a quantity which is comparable to $\|f\|_2^2$ in many cases.

Previously, several other special cases of the conjecture were known. Agmon and Hörmander [AH] studied the case where $d\mu$ is surface measure on a C^1 submanifold and α is an integer, using somewhat smoother cut-off functions. Also, the author [Str1] recently obtained the exact result in the case that $d\mu$ is surface measure on a sphere and $\alpha = n - 1$.

We will say that a measure on $\mathbb{R}^n$ is *locally uniformly α-dimensional* where $0 \leqslant \alpha \leqslant n$ if $\mu(B_r(x)) \leqslant cr^\alpha$ for every ball $B_r(x)$ of radius $r \leqslant 1$. At first this might seem like the wrong definition, because, for example, Lebesgue measure satisfies the definition for all α, $0 \leqslant \alpha \leqslant n$, but we think of Lebesgue measure as being only n-dimensional. However, our definition easily implies that μ is absolutely continuous with respect to α-dimensional Hausdorff measure μ_α. Also, by a simple variant of the Radon–Nikodym theorem proved in Section 3, we can write $\mu = \varphi \, d\mu_\alpha + \nu$, where ν is null with respect to μ_α, in the sense that $\mu_\alpha(E) < \infty$ implies $\nu(E) = 0$. This generalizes the decomposition of an arbitrary measure into discrete and continuous parts, and the measure ν in our decomposition turns out to make a negligible contribution to the limit formula.

If μ is locally uniformly α-dimensional, and $f \in L^2(d\mu)$, then $(f \, d\mu)^\wedge$ is well-defined as a tempered distribution, and in fact it is locally L^2. Our first main results are the estimates

$$\sup_y \sup_{r \geqslant 1} r^{\alpha - n} \int_{B_r(y)} |(f\,d\mu)^\wedge(\xi)|^2\,d\xi \leqslant c\,\|f\|_2^2$$

(Corollary 5.2) and

$$\limsup_{r \to \infty} r^{\alpha - n} \int_{B_r(y)} |(f\,d\mu)^\wedge(\xi)|^2\,d\xi \leqslant c \int |f(x)|^2\,\varphi(x)\,d\mu_\alpha(x)$$

for fixed y (Theorem 5.3). The main ingredient in the proofs is the weak-L^1 estimate for the maximal function

$$m_\alpha f(x) = \sup_{0 \leqslant r \leqslant 1} r^{-\alpha} \int_{B_r(x)} |f|\,d\mu$$

(Corollary 2.5), which is an easy consequence of the Besicovitch covering lemma (Proposition 2.1). In order to get estimates from below we need to make further hypotheses on the measure. Recall that an α-dimensional set E is called regular if the upper and lower α-densities $\limsup_{r \to 0}(2r)^{-\alpha}\,\mu_\alpha(E \cap B_r(x))$ and $\liminf_{r \to 0}(2r)^{-\alpha}\,\mu_\alpha(E \cap B_r(x))$ are equal to one for μ_α-almost every x in E. Because regular sets are so rare (in particular α must be an integer) we define *quasi-regular* sets E by the condition that the lower density be bounded from below on E. For example, the usual Cantor set is quasi-regular. Now if $\mu = \mu_{\alpha|E} + \nu$ is as above and E is regular then we prove

$$\lim_{r \to \infty} r^{\alpha - n} \int_{B_r(y)} |(f\,d\mu)^\wedge(\xi)|^2\,d\xi = c_\alpha \int_E |f|^2\,d\mu_\alpha,$$

while if E is only quasi-regular we have

$$\liminf_{r \to \infty} r^{\alpha - n} \int_{B_r(y)} |(f\,d\mu)^\wedge(\xi)|^2\,d\xi \geqslant c \int_E |f|^2\,d\mu_\alpha$$

(Theorem 5.5). This is the α-dimensional version of Wiener's theorem Notice that it is only for certain regular sets (the conditionthat μ be locally uniformly α-dimensional is not a consequence of regularity) that we obtain the full strength of the naive conjecture for $\mu_{\alpha|E}$ but for quasi-regular sets we come close; for while the limit may not exist (almost certainly it doesn't exist unless E is regular, but we do not prove this), any limit point is equivalent to $\int_E |f|^2\,d\mu_\alpha$.

In Section 6 we give some applications of our results. We prove multiplier theorems for operators from $L^p(d\mu)$ to $L^q(dx)$, and derive properties of Poisson integrals of $f\,d\mu$. We also obtain a restriction theorem for Sobolev spaces. It is well known that functions in the Sobolev space $L^2_\lambda(\mathbb{R}^n)$ have well-defined restrictions in $L^2(M)$ where M is a smooth m-dimensional submanifold of $\mathbb{R}^n$, provided $\lambda > (n-m)/2$. We show that if $\lambda > (n-\alpha)/2$ then functions in $L^2_\lambda(\mathbb{R}^N)$ have well-defined restrictions to any α-dimensional set E for which $d\mu_{\alpha|E}$ is locally uniformly α-dimensional. This result was originally proved by Adams [Ad1, Ad2, Ad3] in slightly sharper form.

In Section 4 we examine the zero-dimensional Wiener theorem in detail. One thing that is rather apparent is that the hypothesis that μ be a finite measure is inappropriate, given the L^2 nature of the conclusion. We remedy this defect in Theorem 4.4. Although this result represents something of a detour from our main goal, the ideas used in the proof form the core of our arguments in Section 5. We also give a version of Wiener's theorem for Hermite expansions—this is a true red herring, but with a twist. Michael Taylor [TM, Section 12.5] has given far-reaching generalizations of Wiener's theorem to the context of spectral expansions of elliptic operators on compact manifolds; our result shows that Hermite expansions behave differently. It seems quite likely that all our results can be generalized to Taylor's context—certainly the reader will encounter no difficulty in replacing Fourier transforms by Fourier series in what follows.

It should be understood that all measures are defined on the Borel sets of $\mathbb{R}^n$ and are assumed locally finite, hence regular. "Measurable set" means "Borel set." For the theory of Hausdorff measure and fractal sets the reader is referred to the excellent book by Falconer [F] which explains the important results clearly, precisely, and concisely.

The results of this paper were announced in [Str2].[1]

Dedication. A. S Besicovitch was one of the most powerful and original mathematicians of our century. He made fundamental and profound contributions to many areas of analysis, including the theory of almost periodic functions and the measure theory of fractal sets (seee [TS] for biography and publication list). I don't know if he ever imagined a connection between these two theories; the connection that I am suggesting here is only vaguely realized, but I hope it would have pleased him.

Personally, I had the great privilege of studying with Besicovitch for a year, when I was too young to be fully aware of his great contributions to mathematics, and he was too old—according to the conventional wisdom—to be still making great contributions to mathematics. How gleefully did "Besie" give the conventional wisdom the thrashing it so richly deserves! With a thick Russian accent, and a twinkle in his eye, he gave us

[1] Further results along these lines are given in "Self-Similar Measures and Their Fourier Transforms."

an inspiring example of the power of the creative mind. Once, after an especially earnest visiting lecturer had delivered a "sermon" on how to teach mathematics to the least gifted students, Besie stood up and related an anecdote from his childhood about his brief experience with a noted violin teacher; when he reached the inevitable conclusion ("and so it was agreed, seeing that I had no talent whatsoever, that the lessons should stop") the audience was in stitches. An important idea that Besie repeatedly emphasized was that a simple argument can still be very deep— it may not be so simple to discover. But above all, what I learned from him, and for which I am eternally grateful, is that good mathematics can be good fun. And it is.

2. Maximal Functions

So much of harmonic analysis begins with maximal functions, and maximal functions are understood via covering lemmas. One of the most powerful covering lemmas is the following, due to Besicovitch (a short proof may be found in de Guzmán [G2, pp. 39–40]). Here $B_r(x)$ denotes the open ball of radius r centered at x. For the result, it is not really necessary that we deal with balls—for example, cubes would do as well, but not general rectangles—but it is essential that the set be centered at x.

Proposition 2.1 (Besicovitch Covering Lemma). *There exists a constant c_n depending only on the dimension, such that if $A \subset \mathbb{R}^n$ is measurable and a collection $\{B_{r(x)}(x)\}_{x\in A}$ of balls centered at each point of A is given with the radii $r(x)$ arbitrary but uniformly bounded, then there exists a finite or countable sub-collection $\{B_k\}$ which covers A with no more than c_n overlaps; i.e.,*

$$\chi_A \leqslant \sum \chi_{B_k} \leqslant c_n \qquad \text{on } \mathbb{R}^n. \tag{2.1}$$

Let μ be any locally finite measure on $\mathbb{R}^n$. (Actually we could do with the following hypothesis: for μ-almost every x there exists $r>0$ such that $0<\mu(B_r(x))<\infty$.) We define the *centered maximal function*

$$M_\mu f(x) = \sup_{r>0} \mu(B_r(x))^{-1} \int_{B_r(x)} |f|\, d\mu \tag{2.2}$$

for any locally integrable f, where we take $0/0=0$ if $\mu(B_r(x))=0$. It is easy to see that $M_\mu f$ is measurable.

Theorem 2.2. *The operator M_μ satisfies the weak-L^1 estimate*

$$\mu\colon \{x : M_\mu f(x) > s\} \leqslant c_n s^{-1} \|f\|_1 \tag{2.3}$$

for all $f \in L^1(d\mu)$, and the L^p estimate

$$\|M_\mu f\|_p \leqslant c_p \|f\|_p \tag{2.4}$$

for all $f \in L^p(d\mu)$, $1 < p \leqslant \infty$, where all L^p norms are with respect to μ.

Proof. Let $E_s = \{x : M_\mu f(x) > s\}$. For every $x \in E_s$ there exists r such that

$$\int_{B_r(x)} |f| \, d\mu \geqslant s\mu(B_r(x)).$$

Assume first that E_s is bounded, so that we may apply the Besicovitch covering lemma to obtain $\{B_k\}$, and then

$$c_n \|f\|_1 \geqslant \sum \int_{B_k} |f| \, d\mu \geqslant \sum s_\mu(B_k) \geqslant s_\mu(E_s)$$

by (2.1), which is (2.3). In the general case we partition $\mathbb{R}^n$ into a countable union of bounded sets, run the above argument above on each bounded set, and then sum. Then (2.4) follows by the Marcinkiewicz interpolation theorem using the trivial $p = \infty$ case. Q.E.D.

This result is also proved in Journé [J]. The next result is proved by different methods by Besicovitch [B1], but also using his covering lemma.

COROLLARY 2.3. *For any $f \in L^1(d\mu)$,*

$$\lim_{r \to 0} \mu(B_r(x))^{-1} \int_{B_r(x)} f \, d\mu = f(x) \tag{2.5}$$

for μ-almost every x and in fact also

$$\lim_{r \to 0} \mu(B_r(x))^{-1} \int_{B_r(x)} |f(y) - f(x)| \, d\mu(y) = 0. \tag{2.6}$$

Proof. Continuous functions are dense in $L^1(d\mu)$ because μ is σ-finite hence regular. Since (2.5) and (2.6) are obviously true for this dense subclass, the result follows for all $L^1(d\mu)$ by general functional analysis principles and the estimate (2.3). Q.E.D.

COROLLARY 2.4. *For any $f \in L^p(d\mu)$, $1 < p < \infty$,*

$$\lim_{r \to 0} \mu(B_r(x))^{-1} \int_{B_r(x)} f \, d\mu = f(x) \quad \text{in} \quad L^p(d\mu). \tag{2.7}$$

Proof. Convergence almost everywhere follows from the previous Corollary (localized), and then L^p convergence follows from (2.4) by the dominated convergence theorem. Q.E.D.

Now fix a real value α satisfying $0<\alpha\leqslant n$, and define the *α-dimensional centered maximal function* by

$$M_\alpha f(x)=\sup_{r>0} r^{-\alpha}\int_{B_r(x)} |f|\, d\mu. \tag{2.8}$$

Similarly we define the local *α-dimensional centered maximal function* by

$$m_\alpha f(x)=\sup_{0<r\leqslant 1} r^{-\alpha}\int_{B_r(x)} |f|\, d\mu.$$

Observe that these maximal functions depend on the measure μ, but this dependence is suppressed in the notation.

We will say that the measure μ is *uniformly α-dimensional* if there exists a constant c such that

$$\mu(B_r(x))\leqslant cr^\alpha \qquad \text{for all} \quad x \text{ and } r>0. \tag{2.9}$$

Similarly, we say that μ is *locally uniformly α-dimensional* if (2.9) holds for $0<r\leqslant$. It is easy to see that a locally uniformly α-dimensional measure must be absolutely continuous with respect to α-dimensional Hausdorff measure μ_α, but such a measure need not exhibit any actual "fractal" behavior. Thus, for example, Lebesgue measure is locally uniformly α-dimensional for any $\alpha\leqslant n$. We can allow $\alpha=0$ in these definitions, in which case a measure is uniformly 0-dimensional if and only if it is finite, and locally uniformly 0-dimensional if and only if $\mu(B_1(x))$ is uniformly bounded in x.

COROLLARY 2.5. *If x is uniformly α-dimensional then M_α is bounded on $L^p(d\mu)$ for $1<p\leqslant\infty$ and satisfies a weak-L^1 estimate, similarly for m_α if μ is locally uniformly α-dimensional.*

Proof. $M_\alpha f\leqslant cM_\mu f$ in the first case, and $m_\alpha f\leqslant cM_\mu f$ in the second case. Q.E.D.

It is also interesting to ask if these results remain true if we drop the requirement that the balls be centered at x, and only require that they contain x. Journé [J] shows that this is the case when the dimension $n=1$, but not when $n\geqslant 2$.

If the measure μ satisfies a doubling condition, then all these results are known [To]. However, most fractal measures do not satisfy a doubling condition.

3. Measure Theory

Let μ be a positive measure with no infinite atoms, not necessarily σ-finite, and let ν be a σ-finite positive measure which is absolutely continuous with respect to μ, $\nu \ll \mu$, in the usual sense ($\mu(E)=0$ implies $\nu(E)=0$). The Radon–Nikodym theorem does not apply in this situation, but there is a simple substitute result. We will say that a measure ν is *null with respect to* μ, written $\nu \lll \mu$, if $\mu(E)<\infty$ implies $\nu(E)=0$. Clearly this is a stronger condition than absolute continuity, and it implies that $\nu(E)=0$ if E is an σ-finite set for μ. In particular, if μ were σ-finite, then only the zero measure could be null with respect to μ. But for non-σ-finite measures μ, such as counting measure on $\mathbb{R}$, it is easy to give examples of non-trivial measures which are null with respect to μ. But again, if $d\nu = f\, d\mu$ for a measurable non-negative function f, then we cannot have ν null with respect to μ unless ν is the zero measure. Thus the null measures and the Radon–Nikodym measures with respect to μ form mutually exclusive classes.

Theorem 3.1. *Let μ be a measure with no infinite atoms, and let ν be σ-finite and absolutely continuous with respect to μ, $\nu \ll \mu$. Then there exists a unique decomposition $\nu=\nu_1+\nu_2$ such that $d\nu_1 = f\, d\mu$ for a non-negative measurable function f, and ν_2 is null with respect to μ, $\nu_2 \lll \mu$.*

Proof. The uniqueness has already been noted. For existence it suffices to consider the case where ν is a finite measure. Then let $\mathscr{A}$ denote the set of measurable sets A such that $\nu(A)>0$ and μ restricted to A is σ-finite. Let a denote the sup of $\nu(A)$ for $A \in \mathscr{A}$, and choose a sequence of sets $A_j \in \mathscr{A}$ such that $\lim_{j\to\infty} \nu(A_j)=a$, and set $B=\bigcup_{j=1}^{\infty} A_j$. We claim $\nu_1 = \nu|_B$ and $\nu_2=\nu|_{cB}$ is the desired decomposition.

Indeed $d\nu_1 = f\, d\mu$ by the Radon–Nikodym theorem since $\mu|_B$ is σ-finite. To show $\nu_2 \lll \mu$ assume $\mu(E)<\infty$. Then $\nu_2(E)=0$ for if not we would have $\nu(B\cup E)>a$ and $B\cup E \in \mathscr{A}$, a contradiction. Q.E.D.

Remark. If μ is counting measure, then the decomposition $\nu=\nu_1+\nu_2$ is just the familiar decomposition of a measure into discrete and continuous parts.

Now we specialize to the case $\mu=\mu_\alpha$, the Hausdorff measure of dimension α on $\mathbb{R}^n$. Recall [F] the definition of the α-upper density

$$\bar{D}_\alpha(\nu, x) = \limsup_{r\to 0}\, (2r)^{-\alpha}\, \nu(B_r(x))$$

of a measure ν. Similarly the α-lower density $\underline{D}_\alpha(\nu, x)$ is defined with the liminf in place of limsup.

THEOREM 3.2. *If ν is a locally finite measure on $\mathbb{R}^n$ that is null with respect to μ_α, $\nu \ll \mu_\alpha$, then $\bar{D}_\alpha(\nu, x) = 0$ for μ_α-almost every x.*

Proof. Let E_k denote the set of $x \in \mathbb{R}^n$ such that for all $\varepsilon > 0$ there exists $r \leqslant \varepsilon$ with $(2r)^{-\alpha} \nu(B_r(x)) \geqslant 1/k$. It is easy to see that the union of the sets E_k is exactly the set of points where $\bar{D}_\alpha(\nu, x) > 0$, so it suffices to show $\mu_\alpha(E_k) = 0$ for every k. We do this first for the case when ν is a finite measure.

Now we apply the Besicovitch covering lemma to the balls whose existence define E_k, and obtan a cover $\{B_{r_j}(x_j)\}$ of E_k such that $\sum \chi_{B_{r_j}}(x_j) \leqslant c_n$ everywhere. However, each ball has radius $r_j \leqslant \varepsilon$, so $B_{r_j}(x_j) \subseteq E_{k,\varepsilon}$ where $E_{k,\varepsilon}$ denotes the set of points of distance $\leqslant \varepsilon$ from E_k. Thus $\sum \chi_{B_{r_j}}(x_j) \leqslant c_n \chi_{E_{k,\varepsilon}}$ hence $\sum \nu(B_{r_j}(x_j)) \leqslant c_n \nu(E_{k,\varepsilon})$. But since we also have $(2r_j)^\alpha \leqslant k\nu(B_{r_j}(x_j))$ we have $\sum (2r_j)^\alpha \leqslant c\nu(E_{k,\varepsilon})$, and letting $\varepsilon \to 0$ this shows $\mu_\alpha(E_k) \leqslant c\nu(E_k)$ by the definition of μ_α and the fact that $E_k = \bigcap_\varepsilon E_{k,\varepsilon}$ and ν is finite. But since ν is finite this means $\mu_\alpha(E_k) < \infty$ hence $\nu(E_k) = 0$ hence $\mu_\alpha(E_k) = 0$.

Finally, if ν is only a locally finite measure, we can apply the same argument to the restriction of ν to any fixed ball B to show $\mu_\alpha(E_k \cap B) = 0$ hence $\mu_\alpha(E_k) = 0$. Q.E.D.

Using the same method of proof, we can give some refinements of Corollaries 2.3, 2.4, and 2.5. We assume now that μ is locally uniformly α-dimensional. It is easy to see that this implies $\mu \ll \mu_\alpha$. Let $\mu = \mu_1 + \mu_2$ be the decomposition of Theorem 3.1, and let E be a set that supports μ_1. (The fact that μ_α contains no infinite atoms follows from a deep theorem of Besicovitch, see below.)

THEOREM 3.3. *For any $f \in L^1(d\mu)$,*

$$\lim_{r \to 0} r^{-\alpha} \int_{B_r(x)} f \, d\mu = 0 \tag{3.1}$$

for μ_α-almost every x in the complement of E.

Proof. We may assume $f \geqslant 0$ and μ is finite, without loss of generality. For each k let $A_k = \{x \notin E$: for all $\varepsilon > 0$ there exists $r \leqslant \varepsilon$ such that $r^{-\alpha} \int_{B_r(y)} f \, d\mu \geqslant 1/k\}$. It suffices to show $\mu_\alpha(A_k) = 0$ for each k, since $\bigcup A_k$ is the subset of the complement of E where (3.1) fails to hold.

Assume first that E supports μ, so $\int_{A_k} f \, d\mu = 0$. We apply the Besicovitch covering lemma to obtain a covering of A_k by balls $\{B_{r_j}(x_j)\}$ such that $\sum \chi_{B_{r_j}}(x_j) \leqslant c_n \chi_{A_{k,\varepsilon}}$. Since $r_j^\alpha \leqslant k \int_{B_{r_j}(x_j)} f \, d\mu$ we have $\sum r_j^\alpha \leqslant kc_n \int_{A_{k,\varepsilon}} f \, d\mu$ which shows $\mu_\alpha(A_k) \leqslant c \int_{A_k} f \, d\mu = 0$.

Now in the general case E supports μ_1, so let E_2 be disjoint from

E and support μ_2. The above argument shows (3.1) holds μ_α-almost everywhere on the complement of $E \cup E_2$, so it suffices to show (3.1) holds μ_α-almost everywhere on E_2. But the above argument also shows $\lim_{r\to 0} r^{-\alpha} \int_{B_r(x)} f\, d\mu_1 = 0$ μ_α-almost everywhere on E_2, so it remains to show $\lim_{r\to 0} r^{-\alpha} \int_{B_r(x)} f\, d\mu_2 = 0$ for μ_α-almost every $x \in E_2$. But this is Theorem 3.2 for $\nu = f\, d\mu_2$. Q.E.D.

We can combine this result with Corollary 2.3 to obtain precise estimates for $\limsup_{r\to 0} r^{-\alpha} \int_{B_r(x)} f\, d\mu$ in case μ is the restriction of μ_α to a set E. We say that a set E is *locally uniformly α-dimensional* if the restriction of μ_α to E is locally uniformly α-dimensional. A powerful theorem of Besicovitch [F, p. 67; B2] shows that every Borel set of infinite μ_α measure contains subsets of arbitrary finite μ_α measure that are locally uniformly α-dimensional. (Besicovitch only proved the result for $F_{\sigma\delta\sigma}$-sets; the extension to Borel sets is due to Davies [D].) In Theorem 5.8 below we show that self-similar fractals are locally uniformly α-dimensional.

COROLLARY 3.4. *Let E be locally uniformly α-dimensional, let μ denote the restriction of μ_α to E, let $f \in L^1(d\mu)$ be non-negative, and set $f(x) = 0$ for $x \notin E$. Then*

$$2^{-\alpha} f(x) \leqslant \limsup_{r\to 0} (2r)^{-\alpha} \int_{B_r(x)} f\, d\mu \leqslant f(x) \tag{3.2}$$

for μ_α-almost every x.

Proof. For $x \notin E$ this is just (3.1). For μ-almost every $x \in E$ we have (2.5) by Corollary 2.3, hence

$$\limsup_{r\to 0} (2r)^{-\alpha} \int_{B_r(x)} f\, d\mu = \bar{D}_\alpha(\mu, x) f(x).$$

The result follows since it is known that $2^{-\alpha} \leqslant \bar{D}_\alpha(\mu, x) \leqslant 1$ for μ-almost every $x \in E$ (see [F, p. 25]—this result is also due to Besicovitch). Q.E.D.

Remarks. In fact it is easy to show that every Borel set E of finite, positive μ_α measure contains locally uniformly α-dimensional subsets E_ε with $\mu_\alpha(E_\varepsilon) \geqslant \mu_\alpha(E) - \varepsilon$ for every $\varepsilon > 0$. Indeed, let

$$F_k = \{x \in E\colon \sup_{0<r\leqslant 1} r^{-\alpha} \mu_\alpha(B_r(x) \cap E) \leqslant k\}.$$

It is easy to see that F_k is measurable and increasing with k, and each F_k is locally uniformly α-dimensional. But μ_α-almost every $x \in E$ belongs to $\bigcup_k F_k$ since $\bar{D}_\alpha(\mu, x) \leqslant 1$ for μ_α-almost every $x \in E$, so $\lim_{k\to\infty} \mu_\alpha(F_k) = \mu_\alpha(E)$. Of course, the constant of local uniform α-dimensionality tends to infinity with k. Nevertheless, the result is interesting because sometimes (see Corollary 5.6) we obtain estimates that are independent of this constant.

These results give us control of $m_\alpha f(x)$ for x outside the support of μ. Indeed if $\limsup_{r\to 0} r^{-\alpha} \int_{B_r(x)} |f|\, d\mu$ is finite then so is $m_\alpha f(x)$, since

$$\sup_{\varepsilon \leqslant r \leqslant 1} r^{-\alpha} \int_{B_r(x)} |f|\, d\mu \leqslant \varepsilon^{-\alpha} \int_{B_1(x)} |f|\, d\mu.$$

Thus if E is as in Corollary 3.4 then $m_\alpha f(x)$ is finite for μ_α-almost every x. More generally, if μ is any locally uniformly α-dimensional measure supported on a set E, then $m_\alpha f(x)$ is finite μ_α-almost everywhere on the complement of E. To see this, assume on the contrary that there exists a set E_1 disjoint from E with $\mu_\alpha(E_1) > 0$ and $m_\alpha f(x) = +\infty$ on E_1. By the above remarks there exists a locally uniformly α-dimensional subset $E_2 \subseteq E_1$ with $0 < \mu_\alpha(E_2) < \infty$. Let $v = \mu_{\alpha|E_2}$ and consider the measure $\mu + v$ and the function f which is extended to be zero on E_2. Clearly $\mu + v$ is locally uniformly α-dimensional, and the maximal function $m_\alpha f$ formed with respect to $\mu + v$ is the same as the one formed with respect to μ. But then Corollary 2.5 applied to $\mu + v$ shows $m_\alpha f$ is finite almost everywhere with respect to v, a contradiction.

4. Wiener's Theorem

We begin with a simple measure theoretic lemma valid for any σ-finite measure μ on a measure space for which points are measurable and with atoms of bounded size. Write $\mu = \mu_1 + \mu_2$ where μ_2 is continuous and $\mu_1 = \sum c_j \delta(x - a_j)$, is discrete.

Lemma 4.1. *Let μ be as above with $c_j \leqslant M$ for all j. Then for any $f \in L^2(d\mu)$ we have*

$$\iint \chi(x = y)\, f(x)\, \overline{f(y)}\, d\mu(x)\, d\mu(y) = \sum |f(a_j)|^2\, c_j^2, \tag{4.1}$$

where $\chi(x = y)$ denotes the characteristic function of the diagonal.

Proof. By Fubini's theorem it suffices to verify the result for one iterated integral. Since $f(x) = f(y)$ whenever $\chi(x = y)$ is different from zero we can write the integral as $\int (\int |f(x)|^2\, \chi(x = y)\, d\mu(y))\, d\mu(x)$. Doing the y integration first we obtain

$$\int |f(x)|^2\, \mu(\{x\})\, d\mu(x)$$

which equals $\sum |f(a_j)|^2\, c_j^2$, and this is finite because $c_j \leqslant M$ and $f \in L^2$.

Q.E.D.

Now let μ be a measure on $\mathbb{R}^n$ which is locally uniformly zero dimensional, meaning

$$|\mu(B)| \leqslant M \tag{4.2}$$

for any ball B of radius one. Clearly this implies that μ is σ-finite and satisfies the hypothesis of Lemma 4.1. It is also easy to verify that if $f \in L^2(d\mu)$ then $f\,d\mu$ is a tempered distribution and so $(f\,d\mu)^\wedge$ is well-defined as a tempered distribution.

LEMMA 4.2. *Under the above hypotheses, $(f\,d\mu)^\wedge \in L^2_{\text{loc}}$, in fact $\int_{\mathbb{R}^n} |(f\,d\mu)^\wedge(\xi)|^2 e^{-t|\xi|^2}\,d\xi < \infty$ for all $t > 0$.*

Proof. By definition

$$\langle (f\,d\mu)^\wedge, \varphi \rangle = \int \hat{\varphi}(x)\,f(x)\,d\mu(x)$$

for any $\varphi \in \mathscr{S}$. Thus to show $(f\,d\mu)^\wedge \in L^2(e^{-t|\xi|^2}\,d\xi)$ it suffices to establish the estimate

$$|\langle (f\,d\mu)^\wedge, \psi(\xi)\,e^{-t|\xi|^2} \rangle| \leqslant c_t \left(\int |\psi(\xi)|^2\,e^{-t|\xi|^2}\,d\xi \right)^{1/2} \tag{4.3}$$

for all $\psi \in \mathscr{S}$. To do this we set $\varphi(\xi) = \psi(\xi)\,e^{-(1/2)t|\xi|^2}$, so that (4.3) becomes

$$|\langle (f\,d\mu)^\wedge, \varphi(\xi)\,e^{-(1/2)t|\xi|^2} \rangle| \leqslant c_t\,\|\varphi\|_2.$$

But we know that

$$(\varphi(\xi)\,e^{-(1/2)t|\xi|^2})^\wedge\,(x) = c_t \int \hat{\varphi}(x-y)\,e^{-(1/2)t^{-1}|y|^2}\,dy$$

so that we need only show

$$\left| \iint \varphi(x-y)\,e^{-t|y|^2}\,dy\,f(x)\,d\mu(x) \right| \leqslant c_t\,\|\varphi\|_2 \tag{4.4}$$

after some trivial changes in notation. We can restate (4.4) as follows: the operator T defined by

$$T\varphi = e^{-t|x|^2} * \varphi$$

is a bounded operator from $L^2(dx)$ to $L^2(d\mu)$.

But now by the Riesz interpolation theorem it suffices to show that T is bounded from $L^1(dx)$ to $L^1(d\mu)$ and from $L^\infty(dx)$ to $L^\infty(d\mu)$. The second statement is trivial, since T maps $L^\infty(dx)$ to continuous bounded functions. For the first, we observe that (4.2) implies

$$\int e^{-t|x-y|^2}\,d\mu(x) \leqslant c_t M, \tag{4.5}$$

and so

$$\int |T\varphi(x)|\,d\mu(x) \leqslant \iint |\varphi(y)|\,e^{-t|x-y|^2}\,dy\,d\mu(x)$$

$$\leqslant c_t M \int |\varphi(y)|\,dy. \qquad \text{Q.E.D.}$$

THEOREM 4.3. *Under the same hypotheses as Lemma* 4.2, *we have*

$$\lim_{t\to 0} t^{n/2} \int |(f\,d\mu)^\wedge(\xi)|^2\,e^{-t|\xi|^2}\,d\xi = \pi^{n/2} \sum |f(a_j)|^2\,c_j^2. \tag{4.6}$$

Proof. A formal calculation shows

$$t^{n/2} \int |(f\,d\mu)^\wedge(\xi)|^2\,e^{-t|\xi|^2}\,d\xi$$

$$= t^{n/2} \iiint f(x)\,\overline{f(y)}\,e^{i(x-y)\cdot\xi}\,e^{-t|\xi|^2}\,d\mu(x)\,d\mu(y)\,d\xi$$

$$= \pi^{n/2} \iint e^{-|x-y|^2/4t}\,f(x)\,\overline{f(y)}\,d\mu(x)\,d\mu(y) \tag{4.7}$$

and as $t \to 0$ the integrand tends to $\chi(x=y)\,f(x)\,\overline{f(y)}$, so that (4.6) would follow from Lemma 4.1, provided we could justify the interchange of limit and integral and the formal computation.

Therefore we begin by looking at

$$\iint e^{-|x-y|^2/4t}\,f(x)\,\overline{f(y)}\,d\mu(x)\,d\mu(y).$$

For $t \leqslant 1/4$ the integrand is dominated by $e^{-|x-y|^2}\,|f(x)\,f(y)|$, and we will show this belongs to $L^1(\mu\times\mu)$. This clearly follows if we can show that the operator S defined by $Sf(x) = \int e^{-|x-y|^2} f(y)\,d\mu(y)$ is bounded on $L^2(d\mu)$. We do this by showing that S is bounded on $L^1(d\mu)$ and $L^\infty(d\mu)$. But both statements are easy consequences of (4.5).

Thus we know that the integral in (4.7) is absolutely convergent, and the dominated convergence theorem applies to establish

$$\lim_{t \to 0} (4\pi)^{-n/2} \iint e^{-|x-y|^2/4t} f(x) \overline{f(y)} \, d\mu(x) \, d\mu(y)$$

$$= (4\pi)^{-n/2} \sum |f(a_j)|^2 c_j^2.$$

Finally to justify (4.7) we note first that if we assume $f \in L^1 \cap L^2(d\mu)$ then all the integrals in (4.7) are absolutely integrable, so (4.7) is valid by Fubini's theorem. For the general $f \in L^2(d\mu)$ we consider the sequence $f_k(x) = f(x) \chi(|x| \leqslant k)$ in $L^1 \cap L^2(d\mu)$ which converges to f in $L^2(d\mu)$. Then

$$\lim_{k \to \infty} (4\pi)^{-n/2} \iint e^{-|x-y|^2/4t} f_k(x) \overline{f_k(y)} \, d\mu(x) \, d\mu(y)$$

$$= (4\pi)^{-n/2} \iint e^{-|x-y|^2/4t} f(x) \overline{f(y)} \, d\mu(x) \, d\mu(y)$$

by the argument above and the dominated convergence theorem, while

$$\lim_{k \to \infty} t^{n/2} \int |(f_k \, d\mu)^\wedge \, (\xi)|^2 e^{-t|\xi|^2} \, d\xi$$

$$= t^{n/2} \int |(f \, d\mu)^\wedge \, (\xi)|^2 e^{-t|\xi|^2} \, d\xi$$

by the proof of Lemma 4.2. Q.E.D.

Remark. The proof also shows

$$\sup_{0 \leqslant t \leqslant 1} t^{n/2} \int |(f \, d\mu)^\wedge \, (\xi)|^2 e^{-t|\xi|^2} \, d\xi \leqslant c \int |f(x)|^2 \, d\mu(x).$$

THEOREM 4.4. *Let v be any complex measure on $\mathbb{R}^n$ satisfying*

$$\sum_{k \in \mathbb{Z}^n} (|v| \, (Q(k)))^2 < \infty, \tag{4.8}$$

where $Q(k)$ denotes the cube of side length 1 centered at k, and write

$$v = \sum c_j \delta(x - a_j) + v_2,$$

where v_2 is continuous. Then $v \in \mathscr{S}'(\mathbb{R}^n)$, $\hat{v} \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ and

$$\lim_{r \to \infty} \frac{1}{\Omega r^n} \int_{|\xi| \leqslant r} |\hat{v}(\xi)|^2 \, d\xi = \sum |c_j|^2, \tag{4.9}$$

where Ω denotes the volume of the unit ball. Furthermore we have

$$\sup_{r\geqslant 1}\frac{1}{r^n}\int_{|\xi|\leqslant r}|\hat{v}(\xi)|^2\,d\xi\leqslant c\sum_{k\in\mathbb{Z}^n}(|v|\,(Q(k)))^2. \tag{4.10}$$

Proof. Define a positive measure μ by $\mu(A)=|v|\,(A)/|v|\,(Q(k))$ for $A\subseteq Q(k)$, so clearly (4.2) satisfied. Furthermore we have $dv=f\,d\mu$ where $|f(x)|=|v|\,(Q(k))$ for $x\in Q(k)$, so $f\in L^2(d\mu)$ by (4.8). Therefore Theorem 4.3 applies to $dv=f\,d\mu$, so

$$\lim_{t\to 0}t^{n/2}\int|\hat{v}(\xi)|^2\,e^{-t|\xi|^2}\,d\xi=\pi^{n/2}\sum|c_j|^2$$

and (4.9) follows by a familiar Tauberian theorem (see [TM]). Finally (4.10) follows from the remark following the proof of Theorem 4.3. Q.E.D.

We may consider

$$\hat{v}(\xi)=\sum c_j e^{ia\cdot\xi}+\hat{v}_2(\xi)$$

as a sum of an almost periodic function and some "noise" $\hat{v}_2(\xi)$, so that Wiener's theorem says that the Bohr mean of $|\hat{v}|^2$ picks out the total energy of the almost periodic component. In Wiener's version, where v is a finite measure we have $\sum|c_j|<\infty$ so the almost periodic component is uniformly almost periodic, and in fact has an absolutely convergent Fourier series. In our version, the restriction on the almost periodic component is that

$$\sum_{k\in\mathbb{Z}^n}\left(\sum_{a_j\in Q(k)}|c_j|\right)^2<\infty \tag{4.11}$$

which is considerably weaker, but not as weak as Besicovitch's B^2 class of almost periodic functions [B3] for which we only need

$$\sum|c_j|^2<\infty. \tag{4.12}$$

However, there are uniformly almost periodic functions which do not satisfy (4.11), essentially because the left side of (4.11) fails to be dilation invariant.

It would appear that the B^2 class of almost periodic functions is the natural class to consider for a generalization of Wiener's theorem of the form: Bohr mean $(|f+\text{noise}|^2)=\sum|c_j|^2$ since Besicovitch shows Bohr mean $(|f|^2)=\sum|c_j|^2$ for

$$f(\xi)\sim\sum c_j e^{ia_j\xi} \tag{4.13}$$

under the assumption (4.12) alone.

We close this section with a brief discussion of the analogue of Wiener's theorem for Hermite and related expansions. Our purpose is to show that there is a fundamental change in the nature of the results. We restrict ourselves to the simplest cases; there are clearly many generalizations possible in the spirit of the other results in this paper.

On $\mathbb{R}^1$ we consider the normalized Hermite functions $h_k(x) = (2^k k!\sqrt{\pi})^{-1/2} e^{-x^2/2} H_k(x)$ where $H_k(x) = (-1)^k e^{x^2}(d/dx)^k e^{-x^2}$ is the kth Hermite polynomial. Then $\|h_k\|_2 = 1$ with respect to Lebesgue measure, and

$$\left(-\frac{d^2}{dx^2} + x^2\right) h_k(x) = (2k+1)\, h_k(x).$$

In fact the system $\{h_k\}_{k=0}^{\infty}$ is the complete eigenfunction system associated with the self-adjoint operator $(-(d^2/dx^2) + x^2)$ on $L^2(\mathbb{R}^1, dx)$ (see [RS]). For any finite measure μ on $\mathbb{R}$, let $\hat{\mu}(k) = \int h_k(x)\, d\mu(x)$, so $\sum_0^\infty \hat{\mu}(k)\, h_k(x)$ is the Hermite expansion for μ.

THEOREM 4.5. *Let* $\mu = \sum c_j \delta(x - a_j) + \mu'$, *where* μ' *is a continuous measure. Then*

$$\lim_{t \to 1^-} (1-t^2)^{1/2} \sum_0^\infty |\hat{\mu}(k)|^2\, t^k = \pi^{-1/2} \sum |c_j|^2 \tag{4.14}$$

and

$$\lim_{N \to \infty} N^{-1/2} \sum_{k=0}^{N-1} |\hat{\mu}(k)|^2 = \sqrt{2}\, \pi^{-1} \sum |c_j|^2. \tag{4.15}$$

Proof. The basic generating function identity for Hermite polynomials is

$$\sum_0^\infty H_k(x)\, H_k(y) \frac{t^k}{2^k k!} = (1-t^2)^{-1/2} \exp\left(\frac{2xyt - (x^2+y^2)t^2}{1-t^2}\right) \tag{4.16}$$

(see [L, p. 61]) for $0 < t < 1$. Therefore

$$\begin{aligned}
&(1-t^2)^{1/2} \sum_0^\infty |\hat{\mu}(k)|^2\, t^k \\
&= \int (1-t^2)^{1/2} \sum_0^\infty t^k h_k(y)\, d\mu(x)\, \overline{d\mu(y)} \\
&= \pi^{-1/2} \int \exp\left(-\left(\frac{x^2+y^2}{2}\right) + \frac{2xyt - (x^2+y^2)t^2}{1-t^2}\right) d\mu(x)\, \overline{d\mu(y)}.
\end{aligned}$$

Now (4 14) follows by the dominated convergence theorem and Lemma 4.1 since the function

$$G_t(x, y) = \exp\left(-\left(\frac{x^2+y^2}{2}\right) + \frac{2xyt - (x^2+y^2)t^2}{1-t^2}\right)$$
$$= \exp\left(-\left(\frac{t}{1-t^2}\right)(x-y)^2 - \frac{1}{2}\left(\frac{1-t}{1+t}\right)(x^2+y^2)\right)$$

is uniformly bounded by one and

$$\lim_{t\to 1^-} G_t(x, y) = \begin{cases} 0 & \text{if } x \neq y \\ 1 & \text{if } x = y. \end{cases}$$

If we set $\varepsilon = 1 - t$ we can rewrite this as

$$\lim_{\varepsilon\to 0^+} \varepsilon^{1/2} \sum |\hat{\mu}(k)|^2 e^{-\varepsilon k} = (2\pi)^{-1/2} \sum |c_j|^2.$$

Then (4.15) follows by a Tauberian theorem [TM, p. 341, Proposition 7.12] with $n = 1$, $l = 1/2$). Q.E.D.

The surprising feature of (4.15) is the power of N that occurs. A similar result holds in $\mathbb{R}^n$.

More generally, we consider the self-adjoint operator $-\Delta + A|x|^\beta$ on $\mathbb{R}^n$, where Δ denotes the Laplacian and $\beta > 1$. Let $\{\varphi_k\}$ denote a complete set of eigenfunctions with eigenvalues $\lambda_1 \leqslant \lambda_2 \leqslant \cdots$ arranged in non-decreasing order. It is known that

$$\lambda_k \sim (ak)^{1/\gamma} \quad \text{as} \quad k \to \infty, \tag{4.17}$$

where

$$\gamma = \frac{n}{2} + \frac{n}{\beta}$$

and

$$a = 2^n A^{n/\beta} \frac{\Gamma(n/2+1)\,\Gamma(\gamma+1)}{\Gamma(n/\beta+1)}.$$

(See [RS, p. 275]. Actually one has the same results for $-\Delta + V$ if $V = |x|^\beta + V_1$ and V_1 is suitably small.)

For μ a finite measure on $\mathbb{R}^n$ write $\hat{\mu}(k) = \int \varphi_k(x)\, d\mu(x)$.

THEOREM 4.6. *Let $\mu = \sum c_j \delta(x - a_j) + \mu'$ where μ' is a continuous measure Then*

$$\lim_{N\to\infty} N^{-\beta/(\beta+2)} \sum_1^N |\hat{\mu}(k)|^2 = b \sum |c_j|^2, \tag{4.18}$$

where

$$b=\left(\frac{2^n\Gamma(\gamma)}{\Gamma(n/\beta+1)}\right)^{\beta/(\beta+2)}\frac{A^{n/(\beta+2)}}{(4\pi)^{n/2}\,\Gamma(n/2+1)^{2/(\beta+2)}}.$$

Proof. Let $K(t_1 x, y)=\sum e^{-t\lambda k}\varphi_k(x)\,\varphi_k(y)$ denote the heat kernel for the operator $-\Delta+A\,|x|^{\beta}$. It is known (see [RS]) that the behavior as $t\to 0$ is the same as the Euclidean heat kernel $(4\pi t)^{-n/2}e^{-|x-y|^2/4t}$, hence

$$\lim_{t\to 0} t^{n/2}\sum e^{-t\lambda k}\,|\hat{\mu}(k)|^2=(4\pi)^{-n/2}\sum|c_j|^2 \tag{4.19}$$

by Lemma 4.1. But then (4.17) and (4.19) imply (4.18) by a Tauberian theorem [TM, p. 341]. Q.E.D.

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5. Main Results

In this section $d\mu$ denotes a measure on $\mathbb{R}^n$ that is locally uniformly α-dimensional as defined in Section 2, for $0\leqslant\alpha\leqslant n$. Note that this implies it is locally uniformly zero dimensional, so Lemma 4.2 applies. For $f\in L^2(d\mu)$ we have $(f\,d\mu)^\wedge\,(\xi)\in L^2_{\text{loc}}$. Many of the results in this section are contained in [AH] in the special case when α is an integer and $d\mu$ is surface measure on a C^1 submanifold of $\mathbb{R}^n$. See also [A].

THEOREM 5.1. *Under the above hypotheses*

$$\sup_{0<t\leqslant 1}\ t^{(n-\alpha)/2}\int e^{-t|\xi|^2}\,|(f\,d\mu)^\wedge\,(\xi)|^2\,d\xi\leqslant c\,\|f\|_2^2. \tag{5.1}$$

Proof. As in the proof of Theorem 4.3, we have

$$t^{(n-\alpha)/2}\int (f\,d\mu)^\wedge\,(\xi)|^2\,e^{-t|\xi|^2}\,d\xi$$

$$=\pi^{n/2}t^{-\alpha/2}\iint e^{-|x-y|^2/4t}f(x)\,\overline{f(y)}\,d\mu(x)\,d\mu(y). \tag{5.2}$$

Now we write

$$t^{-\alpha/2}\int e^{-|x-y|^2/4t}f(x)\,d\mu(x)$$

$$=\int_0^\infty t^{-\alpha/2}e^{-r^2/4t}\,\frac{r}{2t}\int_{|x-y|\leqslant r}f(x)\,d\mu(x)\,dr \tag{5.3}$$

so we can estimate

$$\left|t^{-\alpha/2}\int e^{-|x-y|^2/4t}f(x)\,d\mu(x)\right|$$
$$\leqslant\left(\int_0^\infty t^{-\alpha/2}e^{-r^2/4t}\frac{r}{2t}\mu(B_t(y))\,dr\right)M_\mu f(y).$$

But since we have $|\mu(B_r(y))|\leqslant cr^\alpha$ for $r\leqslant 1$ and trivially $\mu(B_r)\leqslant cr^n$ for $r\geqslant 1$ we have

$$t^{-\alpha/2}\int_0^\infty e^{-r^2/4t}\frac{r}{t}\mu(B_r(y))\,dr$$
$$\leqslant ct^{-1-\alpha/2}\int_0^1 e^{-r^2/4t}r^{1+\alpha}\,dr+ct^{-1-\alpha/2}\int_1^\infty e^{-r^2/4t}r^{n+1}\,dr$$
$$\leqslant c\int_0^\infty e^{-r^2/4}r^{1+\alpha}\,dr+ct^{(n-\alpha)/2}\int_0^\infty e^{-r^2/4}r^2\,dr$$

which is uniformly bounded for $0<t\leqslant 1$. Thus

$$t^{(n-\alpha)/2}\int|(f\,d\mu)^\wedge(\xi)|^2e^{-t|\xi|^2}\,d\xi$$
$$\leqslant c\int M_\mu f(y)\,|f(y)|\,d\mu(y)\leqslant c\,\|f\|_2^2$$

by Theorem 2.2. Q.E.D.

COROLLARY 5.2. *For $d\mu$ and f as above,*

$$\sup_x\sup_{r\geqslant 1}r^{\alpha-n}\int_{B_r(x)}|(f\,d\mu)^\wedge(\xi)|^2\,d\xi\leqslant c\,\|f\|_2^2. \tag{5.4}$$

Proof. Since we can translate $(f\,d\mu)^\wedge$ by multiplying f by an exponential factor that preserves $|f|$, it suffices to consider balls $B_r(0)$ centered at the origin. But then if we choose $t=r^{-2}$ we have $r^{\alpha-n}\leqslant et^{(n-\alpha)/2}e^{-t|\xi|^2}$ for $\xi\in B_r(0)$, so (5.1) implies (5.4). Q.E.D.

Remark. If we assume that μ is uniformly α-dimensional then we can extend the supremum in (5.1) to all $t>0$ and in (5.4) to all $r>0$.

Now if μ is locally uniformly α-dimensional, it is easy to see that μ must be absolutely continuous with respect to μ_α, the α-dimensional Hausdorff measure. Let $\mu=\varphi\,d\mu_\alpha+\nu$ with $\nu\ll\mu_\alpha$ the decomposition of Theorem 3.1.

THEOREM 5.3. *For any $f \in L^2(d\mu)$ we have*

$$\limsup_{t \to 0} t^{(n-\alpha)/2} \int e^{-t|\xi|^2} |(f\,d\mu)^\wedge(\xi)|^2\, d\xi \leqslant c \int |f(x)|^2 \varphi(x)\, d\mu_\alpha(x). \tag{5.5}$$

Also, for fixed y,

$$\limsup_{r \to \infty} r^{\alpha-n} \int_{B_r(y)} |(f\,d\mu)^\wedge(\xi)|^2\, d\xi \leqslant c \int |f(x)|^2 \varphi(x)\, d\mu_\alpha(x), \tag{5.6}$$

where the constant c is independent of y.

Proof. We first estimate

$$\begin{aligned}&\left| t^{-\alpha/2} \int e^{-|x-y|^2/4t} f(x)\, d\nu(x) \right| \\ &\qquad \leqslant \int_0^\infty t^{-\alpha/2} e^{-r^2/4t} \frac{r}{2t} \nu(B_r(y))\, M_\nu f(y)\, dr.\end{aligned}$$

Now by Theorem 3.2, for μ_α-almost every y and every $\varepsilon > 0$ there exists $1 \geqslant \delta > 0$ such that $|\nu(B_r(y))| \leqslant \varepsilon r^\alpha$ for all $r \leqslant \delta$. For such y we break the integral at $r = \delta$, and estimate

$$\begin{aligned}&\int_0^\delta t^{-\alpha/2} e^{-r^2/4t} (r/2t)\, \nu(B_r(y))\, M_\nu f(y)\, dr \\ &\qquad \leqslant \varepsilon \left(\int_0^\infty t^{-1-\alpha/2} e^{-r^2/4t} r^{1+\alpha}\, dr \right) M_\nu f(y) = c\varepsilon M_\nu f(y)\end{aligned}$$

while

$$\begin{aligned}&\int_0^\infty t^{-\alpha/2} e^{-r^2/4t} (r/2t)\, \nu(B_r(y))\, M_\nu f(y)\, dr \\ &\qquad \leqslant \left(c \int_{\delta t^{-1/2}}^{t^{-1/2}} e^{-r^2/4} r^{1+\alpha}\, dr + c t^{(n-\alpha)/2} \int_0^\infty e^{-r^2/4} r^2\, dr \right) \cdot M_\nu f(y) \to 0\end{aligned}$$

as $t \to 0$ as in the proof of Theorem 5.1. Thus for μ_α-almost every y we have

$$\lim_{t \to 0} t^{-\alpha/2} \int e^{-|x-y|^2/4t} f(x)\, d\nu(x) = 0$$

so by the dominated convergence theorem

$$\lim_{t \to 0} t^{(n-\alpha)/2} \int (f\,d\nu)^\wedge(\xi) \overline{(f\,d\mu)^\wedge(\xi)}\, e^{-t|\xi|^2}\, d\xi = 0.$$

Then Theorem 5.1 implies (5.5) and Corollary 5.2 implies (5.6). Q.E.D.

If we regard Corollary 5.2 as a kind of weak Plancherel formula, then by interpolating with the obvious estimate

$$\|(f\,d\mu)^\wedge\|_\infty \leqslant c\,\|f\|_1 \qquad \text{for} \quad f \in L^1(d\mu)$$

we obtain an analogue of the Hausdorff–Young inequality:

COROLLARY 5.4. *If $f \in L^p(d\mu)$ for $1 < p \leqslant 2$ and μ is locally uniformly α-dimensional then*

$$\sup_x \sup_{r \geqslant 1} r^{\alpha - n} \int_{B_r(x)} |(f\,d\mu)^\wedge\,(\xi)|^{p'}\,d\xi \leqslant c\,\|f\|_p^{p'}, \tag{5.7}$$

where $1/p + 1/p' = 1$.

Proof. Apply the Riesz interpolation theorem for each fixed $B_r(x)$ with $r \geqslant 1$ to the estimates

$$\|(f\,d\mu)^\wedge|_{B_r(x)}\|_2 \leqslant c r^{(n-\alpha)/2}\,\|f\|_2$$
$$\|(f\,d\mu)^\wedge|_{B_r(x)}\|_\infty \leqslant \|f\|_1$$

and then take the supremum over x and r. Q.E.D.

If we want to obtain estimates from below, we have to impose further conditions on the measure. Recall that an α-dimensional set E is called *regular* if $\bar{D}(\mu_{\alpha|E}, x) = \underline{D}(\mu_{\alpha|E}, x) = 1$ for μ_α-almost every $x \in E$. However, this condition is extremely restrictive, and can only be satisfied if α is an integer. We will say that E is *quasi-regular* if there exists $\varepsilon > 0$ such that $\underline{D}(\mu_{\alpha|E}, x) \geqslant \varepsilon$ for μ_α-almost every $x \in E$. This is certainly a less restrictive condition, since self-similar fractals are quasi-regular (see Theorem 5.8 below).

THEOREM 5.5. *Let $\mu' = \mu + v$ be a locally uniformly α-dimensional measure on $\mathbb{R}^n$ where $\mu = \mu_{\alpha|E}$ and $v \lll \mu_\alpha$. If E is regular then*

$$\begin{aligned}\lim_{t \to 0} t^{(n-\alpha)/2} &\int e^{-t|\xi|^2}\,|(f\,d\mu')^\wedge\,(\xi)|^2\,d\xi \\ &= \Gamma\left(\frac{n-\alpha}{2} + 1\right) \lim_{r \to \infty} r^{\alpha - n} \int_{B_r(y)} |(f\,d\mu')^\wedge\,(\xi)|^2\,d\xi \\ &= \pi^{n/2} 4^\alpha \Gamma\left(\frac{\alpha}{2} + 1\right) \int_E |f|^2\,d\mu_\alpha \end{aligned} \tag{5.8}$$

for any y. If E is quasi-regular then

$$\liminf_{t \to 0} t^{(n-\alpha)/2} \int e^{-t|\xi|^2}\,|(f\,d\mu')^\wedge\,(\xi)|^2\,d\xi \geqslant c \int_E |f|^2\,d\mu_\alpha \tag{5.9}$$

and also

$$\liminf_{r\to\infty} r^{\alpha-n}\int_{B_r(y)} |(f\,d\mu')^\wedge(\xi)|^2\,d\xi \geqslant c\int_E |f|^2\,d\mu_\alpha, \tag{5.10}$$

where c is independent of y.

Proof. The proof of Theorem 5.3 shows that we may assume $v=0$ without loss of generality. In view of (5.2) and (5.3) we need to understand the behavior of $\int_0^\infty t^{-\alpha/2}e^{-r^2/4t}(r/2t)\,(\int_{B_r(y)} f\,d\mu)\,dr$ for fixed y as $t\to 0$. We split the integral at $r=\delta$, and easily estimate

$$\begin{aligned}
&\lim_{t\to 0}\left|\int_\delta^\infty t^{-\alpha/2}e^{-r^2/4t}\frac{r}{2t}\left(\int_{B_r(y)} f\,d\mu\right)dr\right|\\
&\quad\leqslant \lim_{t\to 0}\int_\delta^\infty t^{-\alpha/2}e^{-r^2/4t}\frac{r}{2t}\,\mu(B_r(y))\,M_\mu f(y)\,dr\\
&\quad\leqslant \lim_{t\to 0} c_\delta\int_\delta^\infty t^{-\alpha/2}e^{-r^2/4t}r^{n+1}\,dr\,M_\mu f(y)\\
&\quad\leqslant \lim_{t\to 0} c_\delta t^{(n-\alpha)/2}\int_0^\infty e^{-r^2/4}r^{n+1}\,dr\,M_\mu f(y)=0.
\end{aligned}$$

If E is regular then for μ_α-almost every $y\in E$ and for every $\varepsilon>0$ there exists $\delta>0$ such that

$$\left|\int_{B_r(y)} f\,d\mu-(2r)^\alpha f(y)\right|\leqslant \varepsilon r^\alpha$$

for all $r\leqslant\delta$. But since

$$\lim_{t\to 0}\int_0^\delta t^{-\alpha/2}e^{-r^2/4t}\frac{r}{2t}(2r)^\alpha f(y)\,dr=4^\alpha\Gamma\left(\frac{\alpha}{2}+1\right)f(y)$$

and

$$\left|\int_0^\delta t^{-\alpha/2}e^{-r^2/4t}\frac{r}{2t}\varepsilon r^\alpha\,dr\right|\leqslant c\varepsilon$$

it follows that

$$\lim_{t\to 0}\int_0^\delta t^{-\alpha/2}e^{-r^2/4t}\frac{r}{2t}\left(\int_{B_r(y)} f\,d\mu\right)dr=4^\alpha\Gamma\left(\frac{\alpha}{2}+1\right)f(y)$$

for μ_α-almost every $y\in E$. This establishes the pointwise convergence of the integrand on the right in (5.2), and since we have already established an integrable upper bound we may apply the dominated convergence theorem to obtain

$$\lim_{t\to 0} t^{(n-\alpha)/2} \int |(f\,d\mu)^\wedge(\xi)|^2 e^{-t|\xi|^2}\,d\xi = \pi^{n/2}4^\alpha \Gamma\left(\frac{\alpha}{2}+1\right)\int |f|^2\,d\mu.$$

The other equality in (5.8) follows from this by the familiar Karamata Tauberian theorem [TM] when $y=0$, and in general since multiplication of f by a complex exponential leaves $|f|$ alone.

Finally suppose E is only quasi-regular. For each fixed y for which $f(y)\neq 0$ and $\varepsilon>0$ there exists $1\geqslant\delta>0$ such that $r\leqslant\delta$ implies

$$\left|\mu(B_r(y))^{-1}\int f\,d\mu - f(y)\right| \leqslant \varepsilon\,|f(y)| \tag{5.11}$$

and

$$\mu(B_r(y))\geqslant cr^\alpha \tag{5.12}$$

by Corollary 2.3 and the definition of quasi-regular. Then

$$\begin{aligned}
&\left|\int_0^\delta t^{-\alpha/2}e^{-r^2/4t}\frac{r}{2t}\left(\int_{B_r(y)} f\,d\mu\right)dr\right.\\
&\qquad\left.-\left(\int_0^\delta t^{-\alpha/2}e^{-r^2/4t}\frac{r}{2t}\mu(B_r(y))\,dr\right)f(y)\right|\\
&\leqslant \varepsilon\int_0^\delta t^{\alpha/2}e^{-r^2/4t}\frac{r}{2t}\mu(B_r(y))\,dr\,|f(y)|\\
&\leqslant c\varepsilon\,|f(y)|.
\end{aligned}$$

Therefore by (5.3) we have

$$\begin{aligned}
&t^{-\alpha/2}\iint e^{-|x-y|^2/4t} f(x)\,\overline{f(y)}\,d\mu(x)\,d\mu(y)\\
&\qquad=\int |f(y)|^2 H(y,t,\varepsilon)\,d\mu(y)+R(t,\varepsilon),
\end{aligned}$$

where

$$H(y,t,\varepsilon)=\int_0^{\delta(y,\varepsilon)} t^{-\alpha/2}e^{-r^2/4t}\frac{r}{2t}\mu(B_r(y))\,dr$$

and the remainder term $R(t,\varepsilon)$ satisfies

$$|R(t,\varepsilon)|\leqslant c\varepsilon\int |f(y)|^2\,d\mu(y).$$

But (5.12) implies

$$\liminf_{t \to 0} H(y, t, \varepsilon) \geqslant c$$

and we have already observed that H is uniformly bounded above, so the dominate convergence theorem implies

$$\liminf_{t \to 0} t^{-\alpha/2} \iint e^{-|x-y|^2/4t} f(x) \overline{f(y)}\, d\mu(x)\, d\mu(y) \geqslant c \int |f(y)|^2\, d\mu(y)$$

if we choose ε small enough. In view of (5.2) this establishes (5.9).

To establish (5.10) we first chop off the tail of the Gaussian. Let λ be a parameter that will eventually be chosen fairly large, and note that when $t\,|\xi|^2 \geqslant \lambda$ we have $e^{-t|\xi|^2} \leqslant e^{-\lambda/2} e^{-(1/2)t|\xi|^2}$ so

$$\begin{aligned} t^{(n-\alpha)/2} &\int_{t|\xi|^2 \geqslant \lambda} e^{-t|\xi|^2}\, |(f\, d\mu)^\wedge\, (\xi)|^2\, d\xi \\ &\leqslant 2^{(n-\alpha)/2} e^{-\lambda/2} \left(\frac{1}{2} t\right)^{(n-\alpha)/2} \int e^{-(1/2)t|\xi|^2}\, |(f\, d\mu)^\wedge\, (\xi)|^2\, d\xi \\ &\leqslant 2^{(n-\alpha)/2} e^{-\lambda/2} c \int_E |f|^2\, d\mu_\alpha \end{aligned}$$

by Theorem 5.3. Thus by taking λ large enough we can improve (5.9) to

$$\liminf_{t \to 0} t^{(n-\alpha)} \int_{t|\xi|^2 \leqslant \lambda} e^{-t|\xi|^2}\, |(f\, d\mu')^\wedge\, (\xi)|^2\, d\xi \geqslant c \int_E |f|^2\, d\mu$$

at the cost of reducing the constant c. By taking $t = \lambda r^{-2}$ and estimating

$$\begin{aligned} r^{\alpha-n} &\int_{B_r(0)} |(f\, d\mu')^\wedge\, (\xi)|^2\, d\xi \\ &\geqslant \lambda^{(\alpha-n)/2} t^{(n-\alpha)} \int_{B_r(0)} e^{-t|\xi|^2}\, |(f\, d\mu')^\wedge\, (\xi)|^2\, d\xi \end{aligned}$$

we obtain (5.10) for $y = 0$. Q.E.D.

Remark. The constants in (5.8) depend on the normalization of Hausdorff measure. In particular, if α is an integer and E is a smooth manifold, then $d\mu_\alpha = (\pi^{\alpha/2}/2^\alpha \Gamma(\alpha/2+1))\, d\sigma_\alpha$ where $d\sigma_\alpha$ is the standard measure on E (induced from the Riemannian metric on E inherited from the embedding of E in $\mathbb{R}^n$). In this case we have

$$\lim_{r \to \infty} r^{\alpha-n} \int_{B_r(y)} |(f\, d\mu')^\wedge\, (\xi)|^2\, d\xi = \frac{2^\alpha \pi^{(n+\alpha)/2}}{\Gamma((n-\alpha)/2+1)} \int_E |f|^2\, d\sigma_\alpha$$

which is consistent with the results in [AH, Str1].

COROLLARY 5.6. *Suppose μ is locally uniformly α-dimensional and $\mu = \mu_{\alpha|E} + \nu$ with $\nu \ll \mu_\alpha$. Then the constants in* (5.5) *and* (5.6) *may be chosen* *to depend only on α, and not on μ. (In particular, the constants do not depend on the constant in* (2.9).)

Proof. By the dominated convergence theorem it suffices to show

$$\limsup_{t\to 0}\left|\int_0^\infty t^{-\alpha/2}e^{-r^2/4t}\frac{r}{2t}\mu(B_r(y))\left(\mu(B_r(y))^{-1}\int_{B_r(y)} f\,d\mu\right)dr\right|\leqslant c\,|f(y)|$$

for μ_α-almost every $y\in E$, in order to control the constant in (5.5). But the proof of the theorem shows that the integral for $r\geqslant\delta$ produces zero in the limit. For μ_α-almost every $y\in E$ we may choose δ so that $\mu(B_r(y))^{-1}\int_{B_r(y)} f\,d\mu$ is arbitrarily close to $f(y)$ for $r\leqslant\delta$. Also, because $\bar{D}_\alpha(\mu_{\alpha|E}, y)\leqslant 1$ for μ_α-almost every $y\in E$, we can choose δ so that

$$\mu(B_r(y))\leqslant(1+\varepsilon)(2r)^\alpha$$

for $r\leqslant\delta$. This gives the desired estimate with $c(\alpha)=\int_0^\infty t^{-\alpha/2}e^{-r^2/4t}(r/2t)(2r)^\alpha\,dr=4^\alpha\Gamma(\alpha/2+1)$. Q.E.D.

It is important to observe that the hypothesis that E is quasi-regular (or even regular) does not imply that $\mu_{\alpha|E}=\mu$ is locally uniformly α-dimensional. For example, when $\alpha=1$, any rectifiable curve E is regular, but for $\mu_{1|E}$ to be locally uniformly 1-dimensional is roughly equivalent to the curve being uniformly Lipschitz. As things stand, we require both hypotheses for Theorem 5.5. One should hope, however, that the hypothesis that E is quasi-regular or regular alone would imply the conclusions of Theorems 5.3 and 5.5. This would require some new idea just to interpret $(f\,d\mu)^\wedge(\xi)$, since it does not follow that $f\,d\mu$ is a tempered distribution. For $\alpha=0$ the theory of Besicovitch almost periodic functions does exactly this. Ultimately, it would be desirable to find some sort of direct characterization of the functions $(f\,d\mu)^\wedge(\xi)$ that arise when $\mu=\mu_{\alpha|E}$ in the way that almost periodicity does when $\alpha=0$.

When E is regular our results given an isometry between Hilbert space norms, hence a polarization argument immediately gives an identity between the associated inner products. However, when E is only quasi-regular this approach yields nothing. Nevertheless, the proof of Theorem 5.3 yields the following Corollary:

COROLLARY 5.7. *Let $\mu=\varphi\,d\mu_\alpha+\nu$ be as in Theorem* 5.3, *and suppose $f, g\in L^p(d\mu)$ have disjoint supports with respect to $\varphi\,d\mu_\alpha$, i.e., $fg\varphi\,d\mu_\alpha=0$ as a measure. Then*

$$\lim_{t\to 0} t^{(n-\alpha)/2}\int e^{-t|\xi|^2}(f\,d\mu)^\wedge(\xi)\overline{(g\,d\mu)^\wedge(\xi)}\,d\xi$$

$$=\lim_{r\to\infty} r^{\alpha-n}\int_{B_r(y)}(f\,d\mu)^\wedge(\xi)\overline{(g\,d\mu)^\wedge(\xi)}\,d\xi=0.$$

Next we show that self-similar fractals satisfy the hypotheses of our theorems. Let $S_1, ..., S_m$ be a finite set of similarity transformations of $\mathbb{R}^n$ with dilation factors $s_1, ..., s_m$ (so that S_j equals s_j times an isometry). The associated dimension α is given by

$$\sum_{j=1}^{m} s_j^\alpha = 1. \tag{5.13}$$

so in particular we are assuming that all the similarities are contractive. We say that a set E is *self-similar* if there is a set of similarities such that $E = \bigcup_{j=1}^m S_j E$, $0 < \mu_\alpha(E) < \infty$, and $\mu_\alpha(S_j E \cap S_k E) = 0$ for $j \neq k$. See [H] or Section 8.3 of [F] for a discussion of this definition and examples. The following result is proved in [H] under slightly stronger hypotheses.

THEOREM 5.8. *Let E be a bounded self-similar set, with α given by* (5.13). *Then E is locally uniformly α-dimensional and quasi-regular.*

Proof. Since $E \supseteq S_j E$ we have $E \cap B_{s_j r}(S_j x) \supseteq S_j(E \cap B_r(x))$ so

$$\frac{\mu_\alpha(E \cap B_{s_j r}(S_j x))}{(2 s_j r)^\alpha} \geqslant \frac{\mu_\alpha(E \cap B_r(x))}{(2r)^\alpha} \tag{5.14}$$

by the basic dilation scaling $\mu_\alpha(sA) = s^\alpha \mu_\alpha(A)$ of Hausdorff measure. This is our key observation, showing that the ratio $\mu_\alpha(E \cap B_r(x))/(2r)^\alpha$ is essentially increasing as r decreases.

To show that E is locally uniformly α-dimensional we will show that $\mu_\alpha(E \cap B_r(x)) \leqslant (2r)^\alpha$ μ_α-almost everywhere on E. Indeed let

$$F_0 = \{x \in E: \mu_\alpha(E \cap B_r(x)) \geqslant (1+\varepsilon)(2r)^\alpha\}$$

for fixed r and ε. Suppose $\mu_\alpha(F_0) > 0$. Define inductively the sets F_k by $F_{k+1} = \bigcup_{j=1}^m S_j F_k$. Then $\mu_\alpha(F_k) = \mu_\alpha(F_0) > 0$ by (5.13), so there exists a set F of positive μ_α-measure such that each $x \in F$ lies in infinitely many F_k. But by (5.14), for each point in F_k there exists a value r_k (equal to r multiplied by k factors chosen from $s_1, ..., s_m$) such that $\mu_\alpha(E \cap B_{r_k}(x)) \geqslant (1+\varepsilon)(2r_k)^\alpha$. Since the value of $r_k \to 0$ as $k \to \infty$ we have $\bar{D}_\alpha(\mu_{\alpha|E}, x) \geqslant (1+\varepsilon)$ for all $x \in F$. This contradicts $\bar{D}_\alpha(\mu_{\alpha|E}, x) \leqslant 1$ μ_α-almost everywhere in E [F, p. 25].

To prove that E is quasi-regular from (5.14) is even easier. Let ε denote a uniform lower bound for the ratio $\mu_\alpha(E \cap B_r(x))/(2r)^\alpha$ as x varies over E and r is restricted to the range $r_1 \leqslant r \leqslant r_2$. If we choose r_1 large enough that $E \cap B_{r_1}(x) = E$ for all $x \in E$ (here we use the boundedness of E) then ε will be positive. We also choose r_2 large enough so that the ratio $r_1/r_2 \leqslant s_j$ for $j = 1, ..., m$. Then for any $r \leqslant r_1$ we use (5.14) inductively to show that for any $x \in E$ there exists $y \in E$ (in fact x is obtained from y by repeated applications of the similarity transformations) such that

$$\frac{\mu_\alpha(E \cap B_r(x))}{(2r)^\alpha} \geqslant \frac{\mu_\alpha(E \cap B_{r'}(y))}{(2r')^\alpha},$$

where $r \leqslant r' \leqslant r_2$. Thus ε is the desired lower bound for $\underline{D}(\mu|_E, x)$. Q.E.D.

Finally, we observe that many of our results can be extended to the case of Hausdorff measures and dimensions associated with more general growth functions, as discussed in Rogers [R]. Let h: $[0, \infty) \to [0, \infty)$ be an increasing, right-continuous function, with $h(t) > 0$ for $t > 0$. Then the Hausdorff measure μ_h can be defined by $\mu_h(E) = \lim_{\delta \to 0} \mu_{h,\delta}(E)$ where $\mu_{h,\delta}(E) = \inf\{\sum_{j=1}^\infty h(\operatorname{diam}(A_j))$: $E \subseteq \bigcup_{j=1}^\infty A_j$ and $\operatorname{diam}(A_j) \leqslant \delta\}$, so that μ_α corresponds to the case $h(t) = t^\alpha$. Clearly the definition only depends on the germ of h at $t = 0$. Similarly we will say that a measure m is *locally uniformly h-dimensional* if $\mu(B_r(x)) \leqslant ch(2r)$ for all x and $r \leqslant 1$, and a set E is *quasi-regular* with respect to μ_h if $\liminf_{r \to 0} \mu_h(B_r(x) \cap E)/h(2r) \geqslant \varepsilon > 0$ for μ_h-almost every $x \in E$.

It is convenient also to assume that h satisfies a *doubling condition*, $h(2t) \leqslant ch(t)$ for all $t \leqslant 1$. Then Theorems 5.1–5.5 remain valid if we substitute $t^{n/2}h(\sqrt{t})^{-1}$ for $t^{(n-\alpha)/2}$ in (5.1), (5.5), (5.8), and (5.9), and if we substitute $(t^n h(r^{-1}))^{-1}$ for $r^{\alpha - n}$ in (5.4), (5.6), (5.7), (5.8), and (5.10). The proofs require only simple modifications. Of course we do not know if there are any new examples of quasi-regular sets in this more general context.

6. Multipliers, Restrictions, and Poisson Integrals

In this section we consider convolution operators, or equivalently, Fourier multipliers, from $L^p(d\mu)$ to $L^q(dx)$, where dx denotes Lebesgue measure and μ is any locally uniformly α-dimensional measure on $\mathbb{R}^n$. (It is clear that such operators cannot be expected to map $L^p(d\mu)$ to $L^q(d\mu)$). We write

$$T_m f = \mathscr{F}^{-1}(m(\xi)(f\,d\mu)^\wedge(\xi)) \tag{6.1}$$

for such an operator, where $m(\xi)$ is a suitable multiplier function. For a different approach to this problem see Adams [Ad1, Ad2, Ad3]; in particular he obtains slightly sharper versions of Corollaries 6.2 and 6.3.

THEOREM 6.1. *Let μ be a locally uniformly α-dimensional measure on $\mathbb{R}^n$ and suppose $1 \leqslant p \leqslant 2 \leqslant q \leqslant \infty$. If $m(\xi)$ satisfies*

$$\left(\int_{|\xi| \leqslant 1} |m(\xi)|^r \, d\xi \right)^{1/r} < \infty, \tag{6.2}$$

where $1/r = 1/p - 1/q$, $|m(\xi)| \leqslant M(|\xi|)$, *where* M *is decreasing and*

$$\int_{1/2}^{\infty} M(x)\, s^{-\alpha/p' + n/q' - 1}\, ds < \infty \tag{6.3}$$

then T_m *is a bounded operator from* $L^p(d\mu)$ *to* $L^q(dx)$ *with the operator norm depending linearly on* (6.2) *and* (6.3).

Proof. By the Hausdorff–Young theorem it suffices to estimate $\|m(\xi)(f\, d\mu)^\wedge(\xi)\|_{q'}$, and for this we use Corollary 5.4. Indeed, by (6.2) and (5.5) for $B_1(0)$ we have

$$\begin{aligned}
&\left(\int_{|\xi| \leqslant 1} |m(\xi)(f\, d\mu)^\wedge(\xi)|^{q'}\, d\xi\right)^{1/q'} \\
&\quad \leqslant \left(\int_{|\xi| \leqslant 1} |m(\xi)|^r\, d\xi\right)^{1/r} \left(\int |(f\, d\mu)^\wedge(\xi)|^{p'}\, d\xi\right)^{1/p'} \\
&\quad \leqslant c\, \|f\|_p,
\end{aligned}$$

so we need only estimate the $L^{q'}$ norm on the region where $|\xi| \geqslant 1$. There we write $M(|\xi|) = -\int_{|\xi|}^{\infty} M'(s)\, ds$ (the hypotheses imply M vanishes at ∞) so

$$|m(\xi)(f\, d\mu)^\wedge(\xi)| \leqslant -\int_1^{\infty} \chi(|\xi| \leqslant s)\, |(f\, d\mu)^\wedge(\xi)|\, M'(s)\, ds.$$

Thus, using (5.5), we obtain

$$\begin{aligned}
&\left(\int_{|\xi| \geqslant 1} |m(\xi)(f\, d\mu)^\wedge(\xi)|^{q'}\, d\xi\right)^{1/q'} \\
&\quad \leqslant -\int_1^{\infty} \|\chi(|\xi| \leqslant s)(f\, d\mu)^\wedge(\xi)\|_{q'}\, M'(s)\, ds \\
&\quad \leqslant -c\int_1^{\infty} \|\chi(|\xi| \leqslant s)(f\, d\mu)^\wedge(\xi)\|_{p'}\, s^{n/r} M'(s)\, ds \\
&\quad \leqslant -c\int_1^{\infty} s^{(n-\alpha)/p'} s^{n/r} M'(s)\, ds\, \|f\|_p \\
&\quad \leqslant c\int_1^{\infty} s^{-\alpha/p' + n/q' - 1} M(s)\, ds\, \|f\|_p + cM(1)\, \|f\|_p.
\end{aligned}$$

Here we have used the fact that M is decreasing to deduce $-M'(s) \geqslant 0$ (in general we must interpret M' as a Stieltjes derivative) and to bound $M(1)$ by the portion of the integral in (6.3) below one. Q.E.D.

In particular, we can apply the theorem to the Bessel potentials

$$J_\lambda f = \mathscr{F}^{-1}((1+|\xi|^2)^{-\lambda/2}(f\,d\mu)^\wedge(\xi)).$$

COROLLARY 6.2. *With μ, p, q as before, J_λ is a bounded operator from $L^p(d\mu)$ to $L^q(dx)$ provided $\lambda > n/q' - \alpha/p'$.*

Another way of stating this result is that $L^p(d\mu) \subseteq L^q_{-\lambda}(\mathbb{R}^n)$ (here $L^p_\alpha(\mathbb{R}^n)$ denotes the Sobolev space equal to the image of $L^p(dx)$ under J_α, with $1 < p < \infty$, [S]).

We can also reinterpret this result as a Sobolev restriction theorem by a simple duality argument. For $f \in L^p_\lambda$ we want to show that there exists a well-defined restriction Rf to the support of μ such that $Rf \in L^q(d\mu)$. To do this it suffices to show

$$\left|\int fg\,d\mu\right| \leqslant c\,\|g\|_{q'}$$

for every $g \in L^{q'}(d\mu)$.

COROLLARY 6.3. *With μ, p, q as before, the restriction Rf is a bounded operator from $L^p_\lambda(\mathbb{R}^n)$ to $L^q(d\mu)$ provided $\lambda > n/p - \alpha/q$. In particular, if $\lambda > (n-\alpha)/2$, then Rf is bounded from $L^2_\lambda(\mathbb{R}^n)$ to $L^2(d\mu)$.*

Proof. For $f \in L^p_\lambda(\mathbb{R}^n)$ write $f = J_\lambda h$ with $h \in L^p(\mathbb{R}^n)$. Then

$$\begin{aligned}\left|\int fg\,d\mu\right| &= \left|\int hJ_\lambda(g\,d\mu)\,dx\right| \\ &\leqslant \|h\|_p\,\|J_\lambda(g\,d\mu)\|_{p'} \\ &\leqslant c\,\|h\|_p\,\|g\|_{q'}\end{aligned}$$

by Corollary 6.2 with (q', p') in place of (p, q). Q.E.D.

We expect that the restriction Rf should also have some smoothness, roughly of order $\lambda - n/p + \alpha/q$. The following result confirms this in a limited way.

THEOREM 6.4. *Let $n=1$, let μ be a finite measure that is uniformly α-dimensional, and suppose $f' \in L^p(dx)$ for some p, $1 \leqslant p < \infty$. Then*

$$\iint \frac{|f(x)-f(y)|^p}{|x-y|^{\alpha+\beta p}}\,d\mu(x)\,d\mu(y) \leqslant c\,\|f'\|_p^p$$

provided $\beta < 1 - (1-\alpha)/p$.

Proof. By the fundamental theorem of calculus and Hölder's inequality we have $|f(x)-f(y)|^p \leqslant |x-y|^{p-1} \int_x^y |f'(s)|^p\, ds$ hence

$$\iint \frac{|f(x)-f(y)|^p}{|x-y|^{\alpha+\beta p}}\, d\mu(x)\, d\mu(y)$$
$$\leqslant \int_{-\infty}^{\infty} |f'(x)|^p \left(\iint_{x<s<y} |x-y|^{p-1-\alpha-\beta p}\, d\mu(x)\, d\mu(y) \right) ds.$$

Since $p-1-\alpha-\beta p > -2\alpha$ by hypothesis it suffices to bound

$$\iint_{x<s<y} |x-y|^{-2\alpha+\varepsilon}\, d\mu(x)\, d\mu(y)$$

uniformly in s for any $\varepsilon>0$.

Because the hypotheses are translation invariant we may take $s=0$. Then for fixed $x<0$ we have

$$\int_0^{\infty} (y-x)^{-2\alpha+\varepsilon}\, d\mu(y) = (2\alpha-\varepsilon) \int_0^{\infty} (y-x)^{-2\alpha+\varepsilon-1}\, \mu([0, y])\, dy$$
$$\leqslant c \int_0^{\infty} (y-x)^{-2\alpha+\varepsilon-1}\, y^{\alpha}\, dy = c\, |x|^{-\alpha+\varepsilon}$$

because μ is assumed uniformly α-dimensional (not just locally). Then

$$\iint_{x<s<y} |x-y|^{-2\alpha+\varepsilon}\, d\mu(x)\, d\mu(y) \leqslant c \int_{-\infty}^{0} |x|^{-\alpha+\varepsilon}\, d\mu(x) \leqslant c$$

because

$$\int_{-1}^{0} |x|^{-\alpha+\varepsilon}\, d\mu(x) = (\alpha-\varepsilon) \int_{-1}^{0} |x|^{-\alpha+\varepsilon}\, \mu([x, 0])\, dx$$
$$+ \mu([-1, 0]) \leqslant c \int_{-1}^{0} |x|^{\varepsilon-1}\, dx + \mu([-1, 0])$$

is finite and $\int_{-\infty}^{-1} |x|^{-\alpha+\varepsilon}\, d\mu(x) \leqslant \mu([-\infty, -1])$ is finite because μ is assumed finite. Q.E.D.

We consider next the behavior of $f\, d\mu$ under convolution with an approximate identity. Let

$$\varphi_t(x) = t^{-n} \varphi(t^{-1}x),$$

where $|\varphi(x)| \leqslant \psi(|x|)$, ψ is decreasing, bounded, and $\int_0^{\infty} r^{n-1} \psi(r)\, dr < \infty$. The Poisson integral is a typical example. We write $u(x, t) = \varphi_t * (f\, d\mu)(x) = \int \varphi_t(x-y)\, f(y)\, d\mu(y)$.

THEOREM 6.5. *Suppose $f \in L^p(d\mu)$ for $1 \leqslant p \leqslant \infty$ where φ is as above and μ is locally uniformly α-dimensional. Then*

$$\left(\int |u(x, t)|^p \, dx \right)^{1/p} \leqslant c t^{(\alpha - n)/p'} \|f\|_p \qquad \text{for} \quad 0 < t \leqslant 1. \tag{6.4}$$

Proof. When $p = 1$, $f \, d\mu$ is a finite measure and (6.4) is well known, since $\varphi(x) \in L^1(dx)$. When $p = \infty$,

$$|u(x, t)| \leqslant \int |\varphi_t(x - y)| \, d\mu(y) \, \|f\|_\infty,$$

so it suffices to show

$$\int |\varphi(t^{-1}(x - y))| \, d\mu(y) \leqslant c t^\alpha. \tag{6.5}$$

However, since

$$|\varphi(t^{-1}(x - y))| \leqslant \psi(t^{-1} |x - y|)$$

and

$$\psi(|x|) = -\int_r^\infty \psi'(r) \, dr$$

with $-\psi' \geqslant 0$ we have

$$\begin{aligned} \int \varphi(t^{-1} |x - y|) \, d\mu(y) &\leqslant -\int_0^\infty \mu(B_{rt}(y)) \, \psi'(r) \, dr \\ &\leqslant -c \int_0^{t^{-1}} (rt)^\alpha \, \psi'(r) \, dr - c \int_{t^{-1}}^\infty (rt)^n \, \psi'(r) \, dr \\ &\leqslant c t^\alpha \int_0^\infty r^{\alpha - 1} \psi(r) \, dr + c t^n \int_0^\infty r^{n-1} \psi(r) \, dr \\ &\leqslant c t^\alpha \qquad \text{for} \quad 0 < t \leqslant 1. \end{aligned}$$

The theorem follows by interpolation. Q.E.D.

Now suppose $\mu = \mu_{\alpha | E} + \nu$ where $\nu \ll \mu_\alpha$. We expect to pick up $f_{|E}$ as a pointwise "limit" of $t^{n-\alpha} u(x, t)$ as $t \to 0$.

THEOREM 6.6. *Suppose $\mu = \mu_{\alpha | E} + \nu$ is locally uniformly α-dimensional, and $f \in L^p(d\mu)$ for $1 \leqslant p \leqslant \infty$. Then*

$$\lim_{t \to 0} t^{n-\alpha} u(x, t) = 0 \tag{6.6}$$

for μ_α-almost every x not in E, while

$$\limsup_{t \to 0} t^{n-\alpha} |u(x, t)| \leqslant c\, |f(x)| \tag{6.7}$$

for μ_α-almost every x in E.

Proof. As before, we have

$$t^{n-\alpha} |u(x, t)| \leqslant -\int_0^\infty r^\alpha \psi'(r)(rt)^{-\alpha} \int_{B_{rt}(x)} |f|\, d\mu\, dr. \tag{6.8}$$

We break the integral at $r = t^{-1}$. For $r > t^{-1}$ we estimate

$$\int_{B_{rt}(x)} |f|\, d\mu \leqslant \|f\|_p\, \mu(B_{rt}(x))^{1/p'} \leqslant c(rt)^{n/p'}\, \|f\|_p$$

with the worst case occurring when $p' = 1$. Thus

$$-\int_{t^{-1}}^\infty r^\alpha \psi'(r)(rt)^{-\alpha} \int_{B_{rt}(x)} |f|\, d\mu\, dr \leqslant c\, \|f\|_p\, t^{n-\alpha} \int_{t^{-1}}^\infty \psi(r)\, r^{n-1}\, dr \to 0$$

as $t \to 0$, so we need only consider the region $r < t^{-1}$. But there the integrand tends to zero for μ_α-almost every x not in E by Theorem 3.3, and is dominated by $-c(x)\, r^\alpha \psi'(r)$ which is integrable, so (6.6) follows by the dominated convergence theorem. Similarly (6.7) follows by the same reasoning from Corollary 3.4. Q.E.D.

THEOREM 6.7. *Suppose φ is a radial decreasing function which is non-negative, bounded and integrable, and μ and f are as in Theorem* 6.6. *If E is regular, then*

$$\lim_{t \to 0} t^{n-\alpha} u(x, t) = \gamma_\alpha(\varphi)\, f(x) \tag{6.9}$$

for μ_α-almost every x in E, where

$$\gamma_\alpha(\varphi) = \alpha 2^\alpha \int_0^\infty \psi(r)\, r^{\alpha-1}\, dr \tag{6.10}$$

and $\psi(r) = \varphi(x)$ for $|x| = r$. If E is only quasi-regular then

$$\liminf_{t \to 0} t^{n-\alpha} |u(x, t)| \geqslant c\, |f(x)| \tag{6.11}$$

for μ_α-almost every x in E.

Proof This time we have an identity

$$t^{n-\alpha}u(x,t) = -\int_0^\infty r^\alpha \psi'(r)(rt)^{-\alpha} \int_{B_{rt}(x)} f\,d\mu\,dr$$

and for the limiting behavior as $t \to 0$ we need only consider the integral for $r < t^{-1}$. If E is regular then

$$\lim_{t\to 0} (rt)^{-\alpha} \int_{B_{rt}(x)} f\,d\mu = 2^\alpha f(x)$$

for μ_α-almost every x, by Corollary 2.3. Then (6.9) follows by the dominated convergence theorem.

If E is only quasi-regular the same argument gives (6.11) if f is non-negative, since

$$\liminf_{t\to 0} (rt)^{-\alpha} \int_{B_{rt}(x)} f\,d\mu = 2^\alpha \underline{D}_\alpha(\mu, x).$$

If f is real-valued write $f = f^+ - f^-$ for the decomposition into positive and negative parts, and write $u^\pm(x,t) = \varphi_t * f^\pm(x)$ (note, however, that $u^\pm$ are not the positive and negative parts of u). At μ_α-almost every x in E where $f(x) > 0$ we have

$$\limsup_{t\to 0} t^{n-\alpha} |u^-(x,t)| = 0$$

by Theorem 6.6, so

$$\liminf_{t\to 0} t^{n-\alpha} |u(x,t)| = \liminf_{t\to 0} t^{n-\alpha} |u^+(x,t)| \geqslant cf^+(x) = c\,|f(x)|$$

by the already established case of (6.11), and a similar analysis with $\pm$ interchanged gives the result when $f(x) < 0$. Finally, if f is complex-valued we obtain the desired result by considering the real and imaginary parts separately. Q.E.D.

An interesting open problem is to characterize the harmonic functions that are Poisson integrals of measures $f\,d\mu$ where $\mu = \mu_{\alpha|E}$.

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